Bosonic string – Kaluza Klein theory exact solutions using 5D–6D dualities

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Abstract

We present the explicit formulae which allow to transform the general solution of the $6D$ Kaluza–Klein theory on a 3–torus into the special solution of the $6D$ bosonic string theory on a 3–torus as well as into the general solution of the $5D$ bosonic string theory on a 2–torus. We construct a new family of the extremal solutions of the $3D$ chiral equation for the $SL(4,R)/SO(4)$ coset matrix and interpret it in terms of the component fields of these three duality related theories.

1 Introduction

In the classical field theory it is possible a situation when the different theories possess the same dynamics. This fact means that one can express the fields of one theory in terms of the fields of another theory by the help of motion equations or without this help. In the first case one deals with the on–shell equivalent theories, whereas in the second case this equivalence has the more general off–shell type.

The string theories [1] provide many examples of the field theories with equivalent dynamics. In the framework of (super)string theory the dynamical equivalence is considered as some type of symmetry, or duality, which acts between the different special string theories as between the different special limits of the single general string theory [2]. However, from the classical point of view, the presence of such a duality means the possibility to obtain new constructive information about one theory from the known information about its dual theory. For example, one can rewrite the exact solution of one theory in the form of exact solution of its dual "partner" using the corresponding algebraical and differential relations which express this duality. The differential relations become necessary in the case of the on–shell equivalence, whereas for the off–shell equivalent theories one needs only in the algebraical calculations.

In this work we present the explicit formulae which establish the off–shell equivalence between the full Kaluza–Klein theory [3] coupled to dilaton and some special subset of the bosonic string theory toroidally compactified to 3 dimensions. The special subset of the bosonic string thery is defined with the zero value of extra Kalb–Ramond and mixed metric field components; the number of the compactified dimensions is left arbitrary. In the special case of the $6D$ theories we show that the pure Kaluza–Klein theory and the corresponding special subset of the bosonic string theory are on–shell equivalent, or dual, to the full $5D$ bosonic string theory and give the explicit formulae which express this duality (see also [4] and references therein). We show that these three theories are equivalent on–shell to the chiral theory with the coset $SL(4, R)/SO(4)$ chiral matrix minimally coupled to the $3D$ General Relativity. We construct a new class of the extremal solutions of this effective $3D$ theory and, using the dual relations, calculate the corresponding solutions of the $6D$ and $5D$ theories under consideration. The constructed solutions have the Majumdar–Papapetrou type [5] and provide a good illustration of the established dualities because they possess the nontrivial values of all the essential duality related field components.

2 σ–model for bosonic string theory

At low energies the bosonic string theory can be described by the field theory of its massless modes. These modes live in D–dimensional space–time and include the dilaton field Φ, the Kalb–Ramond field $B_{MN} = -B_{NM}$ and the metric field $G_{MN} = G_{NM}$. The corresponding action reads [1]:

$$S_D = \int d^D X \sqrt{|\det G_{MN}|} e^{-\Phi} \left(R_D + \Phi_{,M}\Phi^{,M} - \frac{1}{12} H_{MNK} H^{MNK} \right), \tag{1}$$

where $H_{MNK} = \partial_M B_{NK} + \partial_N B_{KM} + \partial_K B_{MN}$.

Below we consider the toroidal compactification of the first d space–time dimensions to the 3–dimensional space, i.e., we put $D = d + 3$ and consider the fields independent on the coordinate X^M with $M = m = 1, ..., d$ (below we denote $y^m \equiv X^m$) and possessing the functional dependence on the coordinates $x^\mu \equiv X^{d+\mu}$ with $\mu = 1, 2, 3$. In this case the field components can be separated in respect to the transformations of the three–dimensional coordinates x^μ in the following way ([6]):

1) two scalar matrices G and B of the dimensions $d \times d$, with the components G_{mk} and B_{mk} correspondingly, and the scalar function

$$\phi = \Phi - \ln\sqrt{|\det G|}; \tag{2}$$

2) two vector columns $\vec{A}_1$ and $\vec{A}_2$ of the dimension $d \times 1$; their components read:

$$\begin{aligned} (\vec{A}_1)_{m\mu} &= (G^{-1})_{mk} G_{k,d+\mu}, \\ (\vec{A}_2)_{m\mu} &= B_{m,d+\mu} - B_{mn}(\vec{A}_1)_{n\mu}; \end{aligned} \tag{3}$$

and

3) two tensor fields

$$\begin{aligned} h_{\mu\nu} &= e^{-2\phi} \left[G_{d+\mu,\,d+\nu} - G_{mk}(\vec{A}_1)_{m\mu}(\vec{A}_1)_{k\nu} \right], \\ b_{\mu\nu} &= B_{d+\mu,\,d+\nu} - B_{mk}(\vec{A}_1)_{m\mu}(\vec{A}_1)_{k\nu} - \frac{1}{2}\left[(\vec{A}_1)_{m\mu}(\vec{A}_2)_{m\nu} - (\vec{A}_1)_{m\nu}(\vec{A}_2)_{m\mu} \right]. \end{aligned} \tag{4}$$

In three dimensions the field $b_{\mu\nu}$ is nondynamical; following [6] one can put $b_{\mu\nu} = 0$ without any contradiction with the motion equations. Moreover, in three dimensions it is possible to

introduce pseudoscalar fields u and v accordingly the relations ([6]–[7])

$$\begin{aligned}
\nabla\times\vec{A}_1 &= e^{2\phi}G^{-1}\left(\nabla u+B\nabla v\right),\\
\nabla\times\vec{A}_2 &= e^{2\phi}G\nabla v-B\nabla\times\vec{A}_1.
\end{aligned}\tag{5}$$

Finally, the bosonic string theory toroidally compactified to three dimensions is equivalent on shell to the effective three–dimensional theory which describes the interacting scalar fields G, B and ϕ and the pseudoscalar ones u and v coupled to the metric $h_{\mu\nu}$. The corresponding action reads:

$$S_3=\int d^3x\sqrt{h}\left\{-R_3+L_3\right\},\tag{6}$$

where

$$L_3=\frac{1}{4}\mathrm{Tr}\left(J_G^2-J_B^2\right)+(\nabla\phi)^2-\frac{1}{2}e^{2\phi}\left[(\nabla u+B\nabla v)^TG^{-1}(\nabla u+B\nabla v)+\nabla v^TG\nabla v\right],\tag{7}$$

$$J_G=\nabla G\,G^{-1},\qquad J_B=\nabla B\,G^{-1},\tag{8}$$

$h=\det h_{\mu\nu}$ and the scalar curvature R_3 is constructed using the metric $h_{\mu\nu}$. This action describes the nonlinear σ–model coupled to the three–dimensional gravity. Below we use this σ–model to describe three different special theories.

3 Duality in arbitrary dimension

Working with the field theory which contains several algebraically independent fields, one can study its special cases when some part of these fields is taken in the form of given functions. For example, one can try to put some fields of the theory equal to their trivial values (zero, for example) and to study the dynamics of the remaining fields. This approach leads to two possible essentially different situations: the resulting dynamics can be free or restricted by the set of additional relations. These relations can arise as the on–shell consequence of our fixation of the fields subset; in the special situation the corresponding restrictions satisfied identically on shell of the remaining field equations. In this "successful" situation we call the set of the remaining fields as the "subsystem" of the original theory.

The effective 3–dimensional theory (6)–(8) possesses the following two subsystems:

I) the subsystem with the trivial (zero) values of the fields B and v; its Lagrangian reads:

$$L_I=(\nabla\phi)^2+\frac{1}{4}\mathrm{Tr}J_G^2-\frac{1}{2}e^{2\phi}\nabla u^TG^{-1}\nabla u.\tag{9}$$

For this subsystem $\vec{A}_2 = 0$, whereas

$$\nabla \times \vec{A}_1 = e^{2\phi} G^{-1} \nabla u; \tag{10}$$

and

II) the subsystem with the trivial (zero) values of the fields B and u with the Lagrangian

$$L_{II} = (\nabla\phi)^2 + \frac{1}{4}\mathrm{Tr} J_G^2 - \frac{1}{2} e^{2\phi} \nabla v^T G \nabla v. \tag{11}$$

Here $\vec{A}_1 = 0$ and

$$\nabla \times \vec{A}_2 = e^{2\phi} G \nabla v. \tag{12}$$

It is easy to see that the map

$$\phi \to \phi, \qquad G \to G^{-1}, \qquad u \to v \tag{13}$$

transforms the subsystem "I" into the subsystem "II". For the vector variables this map gives

$$\vec{A}_1 \to \vec{A}_2. \tag{14}$$

To obtain the physical interpretation of these subsystems one must use the definitions (2)–(5) of the σ–model (7) variables. As the result one concludes that the subsystem "I" corresponds to the full Kaluza–Klein theory coupled to the dilaton field, whereas the subsystem "II" describes the bosonic string theory without extra components of the Kalb–Ramond field and without the mixed ("rotational") components of the metric. The map (13) provides the equivalence of the subsystems "I" and "II"; this equivalence has the off–shell nature. Actually, it is obviously for the effective 3–dimensional theories described by Eqs. (9) and (11); however in fact we consider the physical theories in $d+3$ dimensions. One must come back to the multidimensional fields to see what kind of equivalence really takes place in this case. The map (13)–(14) means the possibility to express the $(d+3)$–dimensional fields for the both subsystems in terms of the same symbols. Let us take the subsytem "I" as the "starting" one; let us denote the matrix G, the rotational vector $\vec{A}_1$ and the multidimensional dilaton Φ for this subsytem as F, $\vec{\omega}$ and Φ correspondingly. Then for the subsystem "I" one has:

$$\begin{aligned} ds_I^2 &= (dy + \vec{\omega} d\vec{x})^T F (dy + \vec{\omega} d\vec{x}) + \frac{e^{2\Phi}}{f} h_{\mu\nu} dx^\mu dx^\nu, \\ \Phi_I &= \Phi, \end{aligned} \tag{15}$$

where

$$f = |\det F|. \tag{16}$$

Below we will need in the column u for the subsystem "I"; let us also put

$$u_I = H. \tag{17}$$

Then, using the map (13)–(14) and Eqs. (2)–(5), for the subsystem "II" one immediately obtains the following expressions for the nonzero field components:

$$\begin{aligned} ds^2_{II} &= dy^T F^{-1} dy + \frac{e^{2\Phi}}{f} h_{\mu\nu} dx^\mu dx^\nu, \\ \Phi_{II} &= \Phi - \ln f, \qquad B_{II\ m,\, d+\mu} = (\vec{\omega})_{m\mu}. \end{aligned} \tag{18}$$

Thus, one can see that our subsystems are off–shell equivalent because Eqs. (15) and (18) are expressed in terms of the same variables using only the algebraical (not differential) operations. In the following section we will see that this fact is not trivial; in this section the subsytem "III", related to the subsystems "I" and "II" via the differential as well as the algebraical relations, will be introduced.

4 Duality between 6D and 5D theories

Let us truncate the subsystems "I" and "II" by putting $\Phi = 0$ in Eqs. (15) and (18). In this case the subsystem "I" describes the pure Kaluza–Klein theory, whereas the subsystem "II" possesses the multidimensional dilaton field uniquely related to the extra metric components. Now we would like to develop some useful matrix formalisms for these simplified subsystems. Namely, for the Kaluza–Klein theory the effective 3–dimensional Lagrangian can be represented in the following (chiral) form [8]:

$$L = \frac{1}{4} \mathrm{Tr}\, \vec{J}^2_M, \tag{19}$$

where $\vec{J}_M = \nabla M\, M^{-1}$. Here the matrices M and M^{-1} are constructed from the σ–model fields F and H as

$$M = \begin{pmatrix} F^{-1} & F^{-1}H \\ H^T F^{-1} & f + H^T F^{-1} H \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} F + f^{-1} H H^T & -f^{-1} H \\ -f^{-1} H^T & f^{-1} \end{pmatrix}. \tag{20}$$

It is easy to see that the matrix M satisfies the $SL(d+1,R)/SO(d+1)$ coset relations

$$\det M = 1, \qquad M^T = M. \tag{21}$$

From the duality (13) it follows that Eq. (19) also gives the effective action for the simplified subsystem "II"; in this case $H \equiv v_{II}$.

Then, the effective 3–dimensional Lagrangian (7) of the bosonic string theory also can be represented in the chiral form [6]–[7]:

$$L_{BS} = \frac{1}{8}\mathrm{Tr}\,\vec{J}^2_{\mathcal{M}_{BS}}, \tag{22}$$

where $\vec{J}_{\mathcal{M}_{BS}} = \nabla \mathcal{M}_{BS}\,\mathcal{M}^{-1}_{BS}$. Here the chiral matrix $\mathcal{M}_{BS}$ reads:

$$\mathcal{M}_{BS} = \left(\begin{array}{cc} \mathcal{G}^{-1} & \mathcal{G}^{-1}\mathcal{B} \\ -\mathcal{B}^T\mathcal{G}^{-1} & \mathcal{G} - \mathcal{B}^T\mathcal{G}^{-1}\mathcal{B} \end{array} \right), \tag{23}$$

and the block components $\mathcal{G}$, $\mathcal{G}^{-1}$ and B are defined as

$$\mathcal{G} = \left(\begin{array}{cc} -e^{-2\phi} + v^T G v & v^T G \\ G v & G \end{array} \right), \quad \mathcal{G}^{-1} = \left(\begin{array}{cc} -e^{2\phi} & e^{2\phi} v^T \\ e^{2\phi} v & G^{-1} - e^{2\phi} v v^T \end{array} \right), \tag{24}$$

$$\mathcal{B} = \left(\begin{array}{cc} 0 & -(u+Bv)^T \\ u + Bv & B \end{array} \right). \tag{25}$$

It is easy to prove that $\mathcal{M}_{BS}$ yields the following coset relations:

$$\mathcal{M}^T_{BS}\mathcal{L}\mathcal{M}_{BS} = \mathcal{L}, \qquad \mathcal{M}^T_{BS} = M_{BS}, \tag{26}$$

where

$$\mathcal{L} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \tag{27}$$

so $\mathcal{M}_{BS} \in O(d+1,d+1)/O(d+1)\times O(d+1)$.

Now let us consider the special case of $d=2$, i.e., the (complete) 5–dimensional bosonic string theory. In [9] it was shown that this theory can be represented by the following chiral matrix of the Kaluza–Klein theory type:

$$M_{BS} = \frac{1}{(\det \mathcal{G})^{\frac{1}{2}}} \left(\begin{array}{cc} \mathcal{G} & \mathcal{G}h \\ h^T\mathcal{G} & \det\mathcal{G} + h^T\mathcal{G}h \end{array} \right), \tag{28}$$

where

$$h_m = \frac{1}{2}\epsilon_{mnk}\mathcal{B}_{nk}. \tag{29}$$

In terms of this matrix the Lagrangian (7) obtains the form (19) of the Kaluza–Klein theory living in 6 dimensions (M_{BS} is the 4×4 matrix):

$$L_{BS} = \frac{1}{4}\mathrm{Tr}\, \vec{J}^2_{M_{BS}}. \tag{30}$$

Using this fact it is possible to identify the $5D$ bosonic string theory with the $6D$ Kaluza–Klein theory, i.e., with the simplified subsystem "I" (and "II") in the case of $d = 3$. The identification relations follow from the comparison of Eqs. (20) and (28):

$$\mathcal{G} = fF^{-1}, \qquad h = H. \tag{31}$$

In the remaining part of this paper we call the (complete) $5D$ bosonic string theory as the "subsytem III". Thus, the subsystem "III" is dual to the 6–dimensional ones "I" and "II". Let us note that the signature $- + + + +$ of the $5D$ space–time of the subsystem "III" corresponds to the one $- - + + ++$ of the $6D$ signature for the subsystems "I" and "II" as it follows from Eqs. (24) and (31).

Now our goal is to establish the type of the equivalence (31) and to express the $5D$ field components of the subsystem "III" in terms of the "physical" off–shell fields F, f, $\vec{\omega}$ and the on–shell defined ones which include H. At the first time let us calculate the scalar field components $e^{\Phi_{III}}$, $e^{2\phi_{III}}$, G_{III} and B_{III}. Using Eqs. (2), (24) and (31), one obtains that

$$e^{\Phi_{III}} = fe^{2\phi_{III}} = -F_{11}, \qquad G_{III} = f(F^{-1})_{22}, \tag{32}$$

where the matrices F and F^{-1} are considered as the block parametrised ones with the components "11", "12", "21" and "22" of the dimensions 1×1, 1×2, 2×1 and 2×2 correspondingly. Then, from Eqs. (25) and (29) it follows that

$$B_{III} = -\sigma_2 H_1, \tag{33}$$

where σ_2 is the 2×2 antisymmetric matrix with $(\sigma_2)_{12} = -1$.

To calculate the vector field components let us consider the motion equation corresponding to Eq. (19):

$$\nabla \vec{J} = 0. \tag{34}$$

From this it follows that it is possible to introduce on–shell the vector matrix $\vec{\Omega}$ accordingly

$$\nabla \times \vec{\Omega} = \vec{J}. \tag{35}$$

Let us parametrise the matrix $\vec{\Omega}$ in the same manner as M; then from Eq. (35) one obtains that $\nabla \times \vec{\Omega}_{21} = f^{-1} F^{-1} \nabla H$, so

$$\vec{\Omega}_{21} = \vec{\omega}. \tag{36}$$

Using the orthogonal coset representation (22)–(23) one can also introduce on–shell the vector matrix Ω_{BS} as

$$\nabla \times \vec{\Omega}_{BS} = \mathcal{G}^{-1} \nabla \mathcal{B} \mathcal{G}^{-1}. \tag{37}$$

Then, using Eqs. (29) and (31) after some algebra one concludes that

$$\vec{\omega}_m = \frac{1}{2} \epsilon_{mnk} (\vec{\Omega}_{BS})_{nk}. \tag{38}$$

Finally, using Eqs. (24), (25), (37) and (38) for the vector field $\vec{A}_{1\ BS}$ one obtains the following result:

$$\vec{A}_{1\ BS} = -\sigma_2 \left(\begin{array}{c} \vec{\omega}_2 \\ \vec{\omega}_3 \end{array} \right). \tag{39}$$

Now let us denote

$$\vec{\Omega}_{11} \equiv -\vec{\omega}^{\star}; \tag{40}$$

then the matrix $\vec{\omega}^{\star}$ satisfies the differential equation

$$\nabla \times \vec{\omega}^{\star} = F^{-1} (\nabla F + f^{-1} \nabla H\, H^T). \tag{41}$$

From this equation and Eq. (5) it follows that

$$\vec{A}_{2\ BS} = \vec{\omega}^{\star}_{21}, \tag{42}$$

where the matrix $\vec{\omega}^{\star}$ is parametrised in the same manner as F. Using Eqs. (3)–(4), (39) and (42) one immediately obtains the algebraical relations for the mixed and the pure 3–dimensional components of the metric and Kalb–Ramond fields; we will not write down them here.

It is easy to see that Eqs. (32), (33), (39), (40)–(42) completely define the duality between the subsystems “I” (or “II”) and “III”. This duality has the on–shell nature because the fields B_{III} and $\vec{A}_{2\ BS}$ are defined in terms of the “Kaluza–Klein” ones using the differential relations. Below we explore the established formulae to construct the duality related solutions for the subsystems “I”–“III”.

5 Special solution and its interpretations

Let us consider the matrix M of the following special form:

$$M = M_0 + \lambda\, CC^T, \tag{43}$$

where

$$M_0 = \mathrm{diag}\,(-1, -1, 1, 1), \tag{44}$$

$\lambda = \lambda(x^\mu)$ is the coordinate function and C is the constant column. Then, the "matter" part (34) of the motion equations as well as their Einstein part

$$R_{3\;mn} = \frac{1}{4}\mathrm{Tr}\left[\left(\vec{J}\right)_\mu\left(\vec{J}\right)_\nu\right]. \tag{45}$$

become satisfied if one supposes that

$$\nabla^2\lambda = 0, \tag{46}$$

$$C^T M_0 C = 0, \tag{47}$$

and h_{mn} corresponds to the flat 3–space. Also from Eq. (47) it follows that $M = M_0 e^{\lambda M_0 CC^T}$, so $\det M = 1$. Let us note, that the matrix M_0 describes the trivial field configuration for the $5D$ bosonic string theory which represents the empty Minkowskian $5D$ space–time with the single time–like coordinate. Thus, one obtains the correct matrix M signature at least in some vicinity of the spatial infinity for any asymptotically trivial solution λ of the Laplace equation (46).

Let us now introduce the functional vector $\vec{\nu}$ on–shell of the solutions of Eq. (46) as

$$\nabla \times \vec{\nu} = \nabla\lambda; \tag{48}$$

and the 3×3 matrix $\Sigma = \mathrm{diag}\,(-1\ -1\ 1)$. Let us also parametrise the column C as

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{49}$$

where c_1 and c_2 are the 3×1 and 1×1 block components correspondingly. Then for the matrices M, $M^{-1} = M_0 - \lambda M_0 CC^T M_0$ and $\vec{\Omega} = \vec{\nu} CC^T M_0$ one obtains:

$$M = \begin{pmatrix} \Sigma + \lambda c_1 c_1^T & \lambda c_2 c_1 \\ \lambda c_2 c_1^T & 1 + \lambda c_2^2 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} \Sigma - \lambda\Sigma c_1 c_1^T\Sigma & -\lambda c_2 \Sigma c_1 \\ -\lambda c_2 c_1^T \Sigma & 1 - \lambda c_2^2 \end{pmatrix}, \tag{50}$$

$$\vec{\Omega} = \vec{\nu} \begin{pmatrix} c_1 c_1^T \Sigma & c_2 c_1 \\ c_2 c_1^T \Sigma & c_2^2 \end{pmatrix}. \tag{51}$$

Then, using Eqs. (20) and (50) one calculates the fields

$$\begin{aligned} f^{-1} &= 1 - \lambda c_2^2, \quad H = \frac{\lambda c_2 \Sigma c_1}{1 - \lambda c_2^2}, \\ F &= \Sigma - \frac{\lambda \Sigma c_1 c_1^T \Sigma}{1 - \lambda c_2^2}, \quad F^{-1} = \Sigma + \lambda c_1 c_1^T, \end{aligned} \tag{52}$$

and also $\vec{\omega} = c_2 c_1 \vec{\nu}$.

Now we are ready to write down the corresponding solutions for the subsystems "I"–"III". Actually, from Eqs. (15)–(16) it immediately follows that the pure Kaluza–Klein solution "I" reads:

$$ds_I^2 = (dy + c_2 c_1 \vec{\nu} d\vec{x})^T \left(\Sigma - \frac{\lambda \Sigma c_1 c_1^T \Sigma}{1 - \lambda c_2^2} \right) (dy + c_2 c_1 \vec{\nu} d\vec{x}) + (1 - \lambda c_2^2) h_{\mu\nu} dx^\mu dx^\nu. \tag{53}$$

Then, the solution for the subsystem "II" has the form

$$\begin{aligned} ds_{II}^2 &= dy^T \left(\Sigma + \lambda c_1 c_1^T \right) dy + (1 - \lambda c_2^2) h_{\mu\nu} dx^\mu dx^\nu, \\ e^{\Phi_{II}} &= 1 - \lambda c_2^2, \quad B_{II\ m, d+\mu} = c_2 (c_1)_m \nu_\mu. \end{aligned} \tag{54}$$

Finally, for the subsystem "III" the developed "technology" (see Eqs. (32), (33), (39), (40)–(42) gives the following result:

$$ds_{III}^2 = -(dy - c_2 \sigma_2 c_1^{''} \vec{\nu} d\vec{x})^T \left(\frac{\sigma_3 - \lambda c_1^{''} c_1^{''\,T}}{1 - \lambda c_2^2} \right) (dy - c_2 \sigma_2 c_1^{''} \vec{\nu} d\vec{x}) + (1 - \lambda c_1^{'\,2}) h_{\mu\nu} dx^\mu dx^\nu,$$

$$e^{\Phi_{III}} = \frac{1 - \lambda c_1^{'\,2}}{1 - \lambda c_2^2}, \quad B = \frac{\lambda c_2 c_1^{'}}{1 - \lambda c_2^2} \sigma_2, \quad B_{m, d+\mu} = \frac{c_1^{'} (c_1^{''})_m \nu_\mu}{1 - \lambda c_2^2}. \tag{55}$$

Here $\sigma_3 = \mathrm{diag}\,(1, -1)$, whereas $c_1^{'}$ and $c_1^{''}$ are the 1×1 and 2×1 block components of the 3×1 column c_1:

$$c_1 = \begin{pmatrix} c_1^{'} \\ c_1^{''} \end{pmatrix}. \tag{56}$$

It is easy to check that $B_{d+\mu,\,d+\nu} = 0$ for the both subsystems “II” and “III” because in these cases the matrix vector functions $\vec{A}_1$ and $\vec{A}_2$ are the products of the corresponding constant matrices and the single functional vector field $\vec{\nu}$. The solutions (53)–(55) describe the extremal field configurations related to the single real harmonic function. In the Einstein–Maxwell theory solutions of this type form the Majumdar–Papapetrou solution class [5]; in the superstring theory they contains the BPS–saturated solutions [4], [10], [11]. Taking the function λ as the sum of the Coulumb–like terms, one obtains for all these theories as well as for our subsystems “I”–“III” the field configurations which describe a set of the arbitrary located point–like interacting sources.

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6 Conclusion

From the formal point of view, the dual theories are equivalent and any result obtained for the one theory can be immediately rewritten as the result for the another theory which belongs to the same “dual multiplet”. In the study of the dual theories it is useful to find the most convenient representation and to use it for the problem solution; the last step is related to the translation of the obtained results into the language of the original physical variables.

In this paper we have studied the “dual triplet” which consists of the 6–dimensional complete Kaluza–Klein theory, 6–dimensional truncated bosonic string theory and 5–dimensional complete bosonic string theory compactified to three dimensions on a torus. In this case the most convenient representation is given by the $6D$ Kaluza–Klein theory chiral formalism. We have established the correspondence formulae (the duality) and have constructed the class of the extremal solutions which are related to the single real harmonic function. In fact it is possible to generalize this class to the case of two real harmonic functions; the corresponding solutions will have the Israel–Wilson–Perjes type [12]. Using the presented general formulae one can map the wide classes of known solutions of the 6–dimensional Kaluza–Klein theory into the 5– and 6–dimensional solutions of the bosonic string theory. However, as it follows from our consideration, one must start from the 6–dimensional Kaluza–Klein theory with

two time–like coordinates to obtain the correct signature in 5 dimensions. Here we will not discuss this formally evident but physically surprising fact.

Our dual theories (the subsytems "I"–"III") form the special cases of the low–energy bosonic string theory. This means that the bosonic string theory (in 7 dimensions at least in view of two necessary time–like dimensions for the $6D$ subsystems) can be considered as some known "generating" theory for our "dual multiplet". The similar situation takes place in the superstring theory where it is known the set of five consistent concrete theories [1] which are related by the web of string dualities. In this case the "generating" theory is named as "M–theory"; now this theory is under construction [2].

References

[1] M.B. Green, J.H. Schwarz, E. Witten, "Superstring theory", Cambridge Univ. Press, 1987.

[2] E. Kiritsis, "Introduction to superstring theory", Lectures presented at the Catholic University of Leuven and University of Padova, 1996-1997. Leuven, Belgium: Leuven Univ. Pr. (1998) 315 p, (Leuven notes in mathematical and theoretical physics. B9).

[3] J.M. Overduin, P.S. Wesson, Phys. Rept. **283** (1997) 303.

[4] D. Yuom, Phys. Rept. **316** (1999) 1.

[5] A. Papapetrou, Proc. Roy. Irish Acad. **A51** (1947) 191; S.D. Majumdar, Phys. Rev. **72** (1947) 930.

[6] A. Sen, Nucl. Phys. **B434** (1995) 179.

[7] A. Herrera Aguilar, O.V. Kechkin, Int. J. Mod. Phys. **A13** (1998) 393;

[8] M. Cvetic, D. Yuom, Phys. Rev. Lett. **75** (1995) 4165.

[9] O.V. Kechkin, M.V. Yurova, Mod. Phys. Lett. **A13** (1998) 219.;

[10] W.A. Sabra, Nucl. Phys. **B510** (1998) 247.

[11] A. Herrera–Aguilar, O.V. Kechkin, Phys. Rev. **D59** (1999) 124006.

[12] W. Israel, G.A. Wilson, J. Math. Phys. **13** (1972) 865; Z. Perjes, Phys. Rev. Lett. **27** (1971) 1668.